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Zero-Dimensional Spectral Measures for Quasi-Periodic Operators with Analytic Potential

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We prove that for quasiperiodic operators with potential $V(n) = \lambda f(\theta + \alpha n)$, f analytic, the spectral measures are zero-dimensional for λ large, any irrational α. It extends a result of Jitomirskaya and Last to the case of any analytic f.

KEY WORDS: Hausdorff dimension; Schrödinger operator; quasi-periodic; spectral measure.

1. INTRODUCTION

In this paper we extend a result of ref. 5. We are working with one-dimensional quasiperiodic Schrödinger operators on $\ell^2(\mathbb{Z})$, defined by

$$(H\psi)(n) = \psi(n+1) + \psi(n-1) + V(n)\,\psi(n) \tag{1.1}$$

The potential is given by $V(n) = \lambda f(\theta + \alpha n)$ where f is a function of period one and throughout α is assumed to be irrational. For the more general setting of ergodic potentials see, e.g., ref. 2.

Herman[(3)] showed that if $f(\theta)$ is a trigonometric polynomial, then for λ sufficiently large, Lyapunov exponents $\gamma(E)$ are positive for any $E \in \mathbb{R}$. Later Sorets and Spencer[(6)] showed that this is true for any analytic f. A different proof has recently been obtained by Bourgain and Goldstein.[(1)] It is known (see, e.g., ref. 2) that positive Lyapunov exponents imply the absence of absolutely continuous spectrum. For $f(\theta)$ a trigonometric polynomial Jitomirskaya and Last have shown that the spectrum is actually of Hausdorff dimension zero when the Lyapunov exponents are positive. Here we will prove the analogous result for any analytic potential, i.e., we show

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Theorem 1. Suppose f is analytic and let μ be the associated spectral measure of H. Then there is λ_0 such that for all $\lambda > \lambda_0$ the following holds: If $\gamma(E)$ is positive for all E in some μ-measurable set A, then the restriction $\mu(A \cap \cdot)$ is zero-dimensional.

Comments. (1) This theorem combined with the Sorets–Spencer result shows that for an analytic potential for λ large enough, the spectral measure is of Hausdorff dimension zero.

(2) The λ_0 appearing in the theorem is no larger than the one needed for the Sorets–Spencer (or Bourgain–Goldstein) proof. So while the result is not optimal in its need for λ large, at least the conclusion of zero dimensionality holds whenever the Lyapunov exponents are known to be positive.

2. PRELIMINARIES

Here we outline the necesary tools of Hausdorff dimension, transfer matrices, Lyupunov exponents, and state the relevant theorem from ref. 5.

For a subset S of $\mathbb{R}$ and $\alpha \in [0, 1]$, the α-dimensional Hausdorff measure, h^α, is given by

$$h^\alpha(S) \equiv \lim_{\delta \to 0} \inf_{\delta-\text{covers}} \sum_{\nu=1}^{\infty} |b_\nu|^\alpha \tag{2.1}$$

where a δ-cover is a cover of S by a countable collection of intervals, $S \subset \bigcup_{\nu=1}^{\infty} b_\nu$, such that for each ν the length of b_ν is at most δ. Given that $S \neq \emptyset$, there exists an $\alpha(S) \in [0, 1]$ such that $h^\alpha(S) = 0$ for any $\alpha > \alpha(S)$, and $h^\alpha(S) = \infty$ for any $\alpha < \alpha(S)$. This unique $\alpha(S)$ is called the Hausdorff dimension of S. A measure is called zero-dimensional if it is supported on a set of Hausdorff dimension zero.

We denote the n-step transfer-matrix of

$$Hu = Eu \tag{2.2}$$

by $\Phi_n(E)$:

$$\Phi_n(E) \equiv M_n(E)\, M_{n-1}(E) \cdots M_1(E)$$

where

$$M_n(E) \equiv \begin{pmatrix} E - V(n) & -1 \\ 1 & 0 \end{pmatrix} \tag{2.3}$$

In our case, $V(n)=f(T^n\theta)$ where T is the ergodic transformation $T(\theta)=\theta+\alpha$, the corresponding operator H and therefore the transfer-matrices Φ_n and M_n will depend on θ. We have $M_n(E,\theta)=M(E,T^n\theta)$, where

$$M(E,\theta)\equiv\begin{pmatrix} E-\lambda f(\theta) & -1 \\ 1 & 0 \end{pmatrix} \tag{2.4}$$

The Lyapunov exponent, $\gamma(E)$, is defined as

$$\gamma(E)\equiv\lim_{k\to\infty}\frac{\int_X \ln\|\Phi_k(E,\theta)\|\,d\mu(\theta)}{k}=\inf_k\frac{\int_X \ln\|\Phi_k(E,\theta)\|\,d\mu(\theta)}{k} \tag{2.5}$$

For any θ the upper Lyapunov exponent $\bar{\gamma}(E)$ is defined by

$$\bar{\gamma}(E)\equiv\limsup_{n\to\infty}\,(1/n)\ln\|\Phi_n(E)\|$$

In ref. 4 it is shown that for a half-line operator if $\bar{\gamma}>0$ for all E in some μ-measurable set A, the μ restricted to A is zero-dimensional. To state the lemma from ref. 5 on how to extend this to a whole line operator we introduce some notation. We define the "right" and "left" transfer matrices by $\Phi_n^+(E)\equiv\Phi_n(E)$, and $\Phi_n^-(E)\equiv M_{-n+1}(E)\,M_{-n+2}(E)\cdots M_{-1}(E)\,M_0(E)$.

The result we use is

Lemma 2 (ref. 5). Suppose that for every E in some μ-measurable set A there exist $a,c>0$ and sequences $k_n\to\infty$, $j_n^{\pm}\leqslant ck_n$, such that $\|\Phi^{\pm}_{j_n^{\pm}}\|\geqslant e^{ak_n}$. Then the restriction $\mu(A\cap\cdot)$ is zero-dimensional.

3. PROOF OF THEOREM 1

For now we leave out the "plus or minus" notation. Assume we are dealing with, say, the "right" transfer matrices. At the end we point out how to relate the two different matrices to be able to apply Lemma 2.

We start by using a product from ref. 6 to obtain an important argument calculation (3.6). Since f is periodic and analytic we may, slightly abusing notations, consider $f(z)=f(e^{2\pi i\theta})\equiv f(\theta)$ to be defined on a neighborhood of $|z|=1$. Let

$$A_k=A_k(z)=E-\lambda f(z_k)$$

where $z_n=e^{2\pi in\alpha}\cdot z$. Denote the top left of $\Phi_n(E)$ by g_n. Now set $\eta_1=0$ and

$$\eta_{k+1}=1/(A_k-\eta_k)$$

Then

$$\Phi_n(E)\begin{pmatrix}1\\ \eta_1\end{pmatrix}=\prod_{k=1}^{n}(A_k-\eta_k)\cdot\begin{pmatrix}1\\ \eta_{n+1}\end{pmatrix}$$

and so

$$g_n=\prod_{k=1}^{n}(A_k-\eta_k) \tag{3.1}$$

Now we use a fact which seems to be implicit in ref. 6 and is proven explicitly in ref. 1. The fact is: There is $m_0>0$ such that for any real E_1 there is a radius $r_1<1$ such that for all $|z|=r_1$, $|f(z)-E_1|>m_0$. By reflection we get the same for $r_2>1$. Using this with $E_1=E/\lambda$ we get that $|f-E/\lambda|>m_0$ on $|z|=r_1, r_2$, or, as we use,

$$|\lambda f-E|>\lambda m_0 \tag{3.2}$$

Note that for $\lambda m_0>2$ we have

$$|\eta_k|<1 \tag{3.3}$$

for all k and so

$$|g_n(z)|=\prod_{k=1}^{n}|A_k-\eta_k|\geqslant\prod_{k=1}^{n}(|A_k|-1)>(\lambda m_0-1)^n \tag{3.4}$$

Take t such that $e^{n\gamma/t}<(\lambda m_0-1)^n$. From (3.4) we get that on $|z|=r_1, r_2$

$$|g_n-(g_n-e^{n\gamma/t})|<|g_n| \tag{3.5}$$

Recall that for a contour Γ the argument of a funcion f over Γ is

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f'}{f}\,dz$$

Let Γ be the contour which goes around $|z|=r_1$ and $|z|=r_2$ clockwise and counter-clockwise irrespectively. Then with the argument always over Γ, (3.5), (3.1), (3.2), (3.3) and Rouche's Theorem imply

$$\begin{aligned}\operatorname{Arg}(g_n-e^{n\gamma/t})&=\operatorname{Arg}(g_n)=\operatorname{Arg}\prod_{k=1}^{n}(A_k-\eta_k)\\&=\prod_{k=1}^{n}\operatorname{Arg}(A_k-\eta_k)=\sum_{k=1}^{n}\operatorname{Arg}A_k=n\cdot\operatorname{Arg}(\lambda f-E)\end{aligned} \tag{3.6}$$

Note that the boundedness of f immediately implies

$$\|\Phi_n(\theta, E)\| \leqslant e^{nC} \tag{3.7}$$

for some $C < \infty$.

Let $A_k = \{\theta \in [0, 1) : \|\Phi_k(\theta, E)\| > e^{k\gamma(E)/t}\}$. Since $\gamma(E) = \inf_k k^{-1} \int_0^1 \ln \|\Phi_k(\theta, E)\| \, d\theta$, $k\gamma(E) \leqslant \int_0^1 \ln \|\Phi_k(\theta, E)\| \, d\theta = \int_{A_k} + \int_{[0,1)\setminus A_k} \leqslant |A_k| \, kC + (1 - |A_k|) \, k\gamma(E)/t$, where $|\cdot|$ stands for Lebesque measure. So

$$|A_k| \geqslant \frac{k\gamma(E) - k\gamma(E)/t}{kC - k\gamma(E)/t} = \frac{\gamma(E)(1 - 1/t)}{C - \gamma(E)/t} \equiv c(E)$$

By the above argument calculation $\{\theta \in [0, 1) : |g_k| > e^{k\gamma(E)/t}\} \subseteq A_k$ consists of no more than k Arg$(\lambda f - E)$ intervals. Therefore there exists a segment, $\Delta_k \subset A_k$, with $|\Delta_k| \geqslant c(E)/(k \text{ Arg})$.

We will now show that for every θ there exists a sequence $j_n \to \infty$ such that $\|\Phi_{j_n}\|$ is exponentially large.

This sequence will be related to a continued fractions expansion of α. Let p_n/q_n be the sequence of continued fraction approximants of α. Let $\Delta \subset [0, 1)$ be an arbitrary segment. We use the following from ref. 5.

Lemma 3. Let $n \in \mathbb{N}$ be such that $|\Delta| > 1/q_n$. Then for any θ there exist k in $\{0, 1, ..., q_n + q_{n-1} - 1\}$ such that $T_\alpha^k \theta \in \Delta$.

Now set

$$k_n = \left[\frac{c(E) \, q_n}{\text{Arg}}\right] + 1 \tag{3.8}$$

By the lemma for every θ there exists j in $\{0, 1, ..., q_n + q_{n-1} - 1\}$ such that $T_\alpha^j \theta \in A_{k_n - 1}$, and so $\|\Phi_{k_n - 1}(T_\alpha^j \theta)\| > e^{(k_n - 1)\gamma(E)/t}$. Since $\Phi_{j + k_n - 1}(\theta) = \Phi_{k_n - 1}(T_\alpha^j \theta) \, \Phi_j(\theta)$, and each Φ_i is unimodular we obtain that either $\|\Phi_j(\theta)\|$ or $\|\Phi_{j + k_n - 1}(\theta)\|$ is greater than $e^{((k_n - 1)\gamma(E))/2t}$. Let

$$j_n = \min\{j \in \{0, ..., q_n + q_{n-1} - 1 + k_n - 1\} : \|\Phi_j(\theta)\| \geqslant e^{((k_n - 1)\gamma(E))/2t}\} \tag{3.9}$$

To finish the proof it now remains to notice that the two inequalities

$$\|\Phi_j(\theta)\| \geqslant e^{((k_n - 1)\gamma(E))/2t} \geqslant e^{[\gamma(E)/4t] \cdot [(c(E) \, q_n)/\text{Arg}]}$$

and

$$j_n \leqslant q_n + q_{n-1} - 1 + k_n \leqslant q_n + q_{n-1} + \frac{c(E) \, q_n}{\text{Arg}} \leqslant q_n \left[2 + \frac{c(E)}{\text{Arg}}\right]$$

together imply

$$\|\Phi_{j_n}(\theta)\| \geqslant e^{j_n \cdot \text{const}}$$

where const $= [(\gamma(E)\, c(E))/(4t \text{ Arg})] \cdot [1/(2 + c(E)/\text{Arg})]$.

Now we point out how to apply Lemma 2. In our (quasiperiodic) case we have

$$\Phi_n^{+,\alpha}(\theta, E) = M(T_\alpha^n \theta, E)\, M(T_\alpha^{n-1}\theta, E) \cdots M(T_\alpha \theta, E)$$

and

$$\Phi_n^{-,\alpha}(\theta, E) \equiv M(T_{-\alpha}^{n-1}\theta, E)\, M(T_{-\alpha}^{n-2}\theta, E) \cdots M(T_{-\alpha}\theta, E)\, M(\theta, E)$$

and

$$\Phi_n^{-,\alpha}(\theta, E) = \Phi_n^{+,(-\alpha)}(\theta + \alpha, E) \tag{3.10}$$

Fix $\varepsilon > 0$. Note that the denominators q_n are the same for α and $-\alpha$. So from (2.5), (3.9) and (3.10) we obtain that, for n sufficiently large, the conditions of Lemma 2 are satisfied with $A = \{E: \gamma(E) > 0\}$, $a =$ const, $c = [\text{Arg}/c(E)][2 + c(E)/\text{Arg}]$, k_n defined by (3.8), and sequences $j_n^{\pm}$ defined by (3.9) with $\Phi_n = \Phi_n^{\pm,\alpha}$.

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